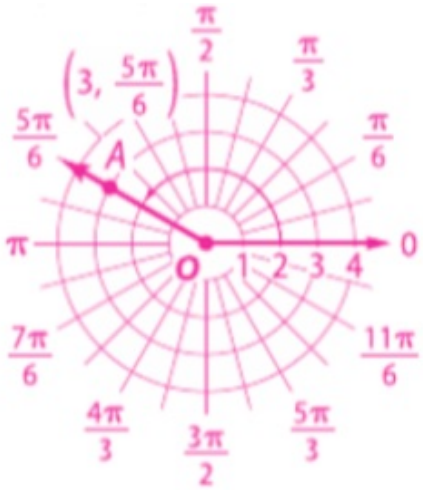
❖ Topic 1 Polar Coordinates and Complex Numbers

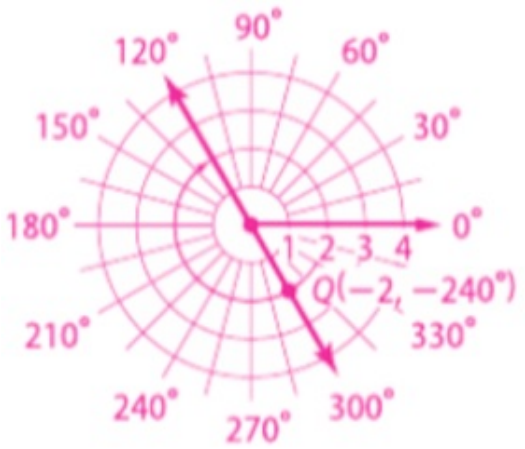
Subtopic 1.1 Polar Coordinates

1. Graph each point on a polar grid.

$A(3,\frac{5\pi}{6})$

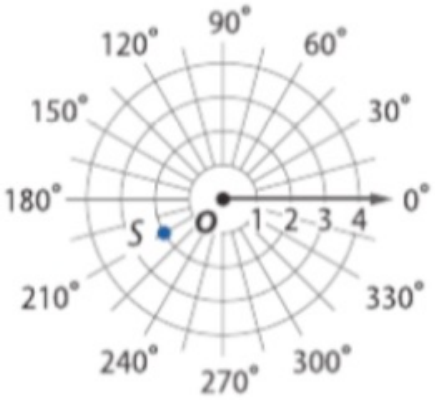


$A(-2,-240°)$



2. Find four different pairs of polar coordinates that name point S if $-360° < \theta < 360°$

$(2,-150°), (2,210°), (-2,30°), (-2,-330°)$



3. Which of the following Does Not represent another pair of polar coordinates for point A $(3,120°)$?

✓ A. $(3,-300°)$

B. $(-3,-60°)$

C. $(3,-240°)$

D. $(3,480°)$

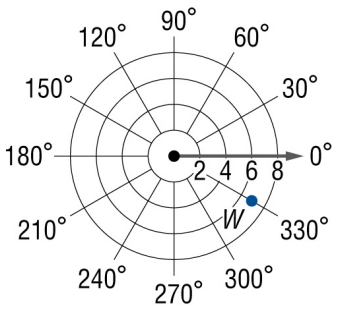
4. Find four different pairs of polar coordinates that name point W if $-360^0 \leq \theta \leq 360^0$.

A. (7, –30), (7, 150°), (7, 210°), (7, 330°)

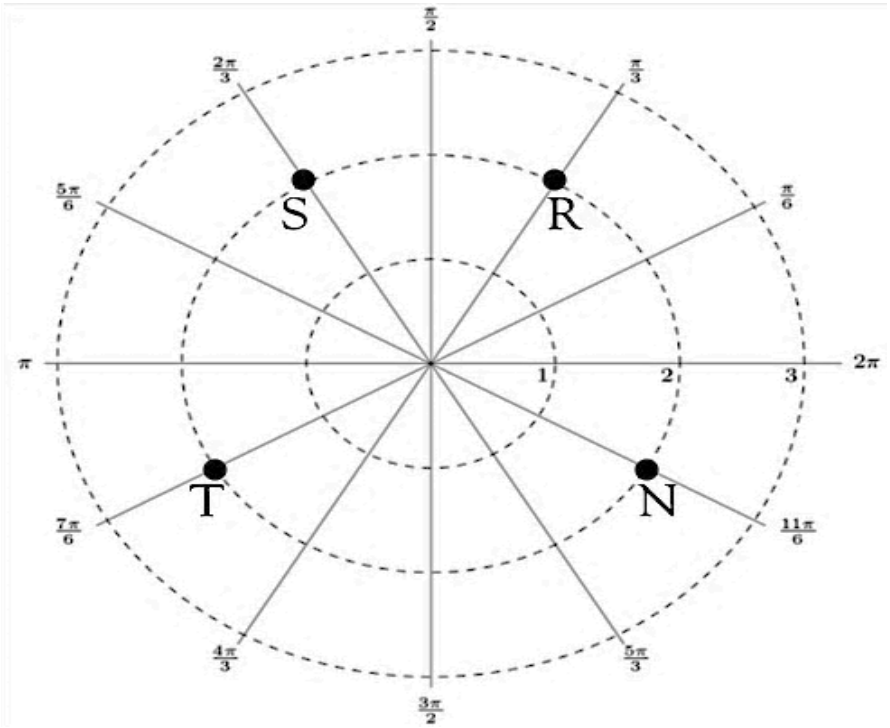
B. (5, –30°), (5, 330°), (–5, 150°), (–5, -210°)

✓ C. (7, –30°), (7, 330°), (–7, 150°), (–7, -210°)

D. (7, –150°), (7, 330°), (–7, 30°), (–7, 210°)



5. Given the below grid. Answer questions (i-iii).



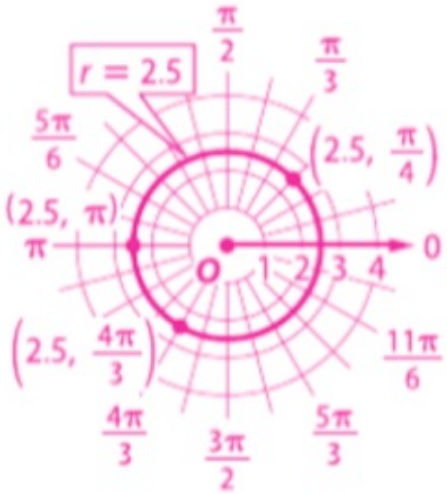
i. The point with the polar coordinates $(-2, 240°)$ on the given polar grid, is __R___.

ii. The point with the polar coordinates $(-2, 300°)$ on the given polar grid, is __S___.

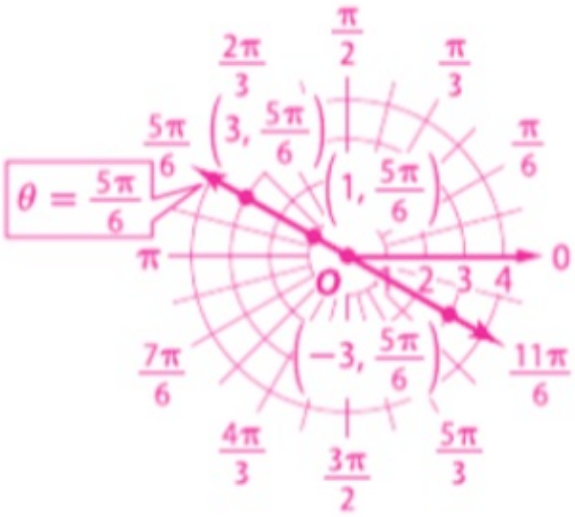
iii. The point with the polar coordinates $(-2, 30°)$ on the given polar grid, is __T___.

6. Graph each polar equation.

$r = 2.5$



$\theta = \frac{5\pi}{6}$



7. Find the distance P_1P_2 between each pair of points to the nearest tenth degree.

$P_1(3,40°), P_2\ (5,130°)$

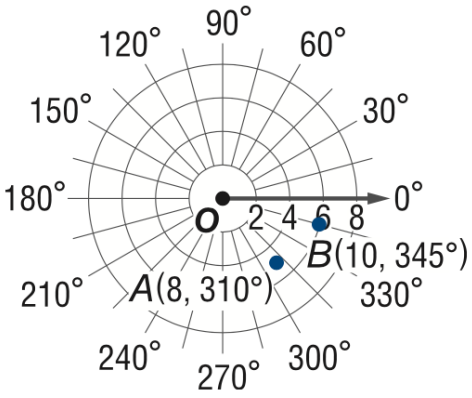
$P_1P_2 = \sqrt{9 + 25 - 2(3)(5)\cos(130 - 40)}$
$P_1P_2 = \sqrt{34 - 30\cos(90)}$
$P_1P_2 = \sqrt{34 - 30(0)}$
$P_1P_2 = \sqrt{34}$=5.8

8. A naval radar is tracking two aircrafts. The coordinates of the two carriers are (8,150°)and (3,65°) where r is measured in kilometers.
What is the distance between the two aircraft carriers.

$d = \sqrt{(8)^2 + (3)^2 - 2(8)(3)\cos(65 - 150)} \approx 8.3$ kilometers

9. An air traffic controller is tracking two airplanes that are flying at the same altitude. The coordinates of the planes are A(8, 310°) and B(10, 345°), where the directed distance is measured in kilometers.

   i. Sketch the graph of the situation.



   ii. If regulations prohibit airplanes from passing within five kilometers of each other, are these airplanes in violation? Explain.

   Using the polar distance formula, we have:

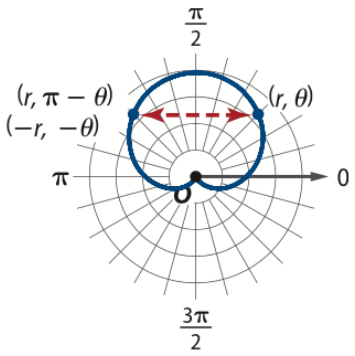
   $\text{AB} = \sqrt{8^2 + 10^2 - 2(8)(10)\cos(345 - 310)} \approx 5.65\ km$

The planes are about 5.65 kilometers apart, so they are not in violation of this regulation.

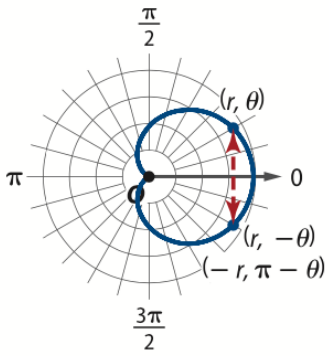
Subtopic 1.2 Graphs of Polar Equations

KeyConcept Symmetry of Polar Graphs

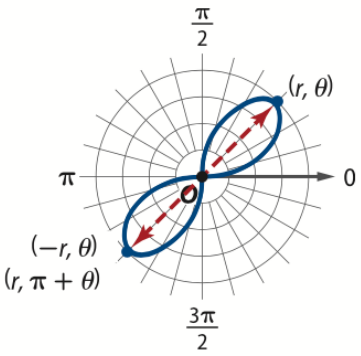
Symmetry with Respect to the Line $\theta = \frac{\pi}{2}$



Symmetry with Respect to Polar Axis



Symmetry with Respect to the Pole



1. Apply the symmetry tests to prove that the graph $r = 4sin3\theta$ is symmetric about $\theta = \frac{\pi}{2}$. (y-axis)

   Replace r by $-r$ and θ by $-\theta$

   $-r = 4\sin(-3\theta)$

   $-r = -4sin3\theta$

   $r = 4sin3\theta$

   Because the resulting equation is equivalent to the original equation, then the graph of this equation is symmetric with respect to $\theta = \frac{\pi}{2}$ or the y-axis.

2. Apply the symmetry tests to prove that the graph $r = 1 + 2cos\theta$ is symmetric about the polar axis (x-axis).

   Replace θ by $-\theta$

   $r = 1 + 2\cos(-\theta)$

   $r = 1 + 2cos\theta$

   Because the resulting equation is equivalent to the original equation, then the graph of this equation is symmetric with respect to the polar axis (x-axis).

3. Apply the symmetry tests to prove that the graph $r^2 = 15 \sin 2\theta$ is symmetric about the pole (the origin).

Replace r by $-r$

$(-r)^2 = 15 \sin 2\theta$

$r^2 = 15 \sin 2\theta$

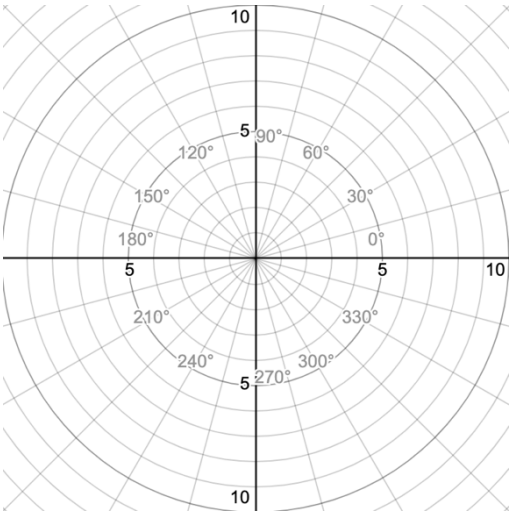
Because the resulting equation is equivalent to the original equation, then the graph of this equation is symmetric with respect to the pole (the origin).

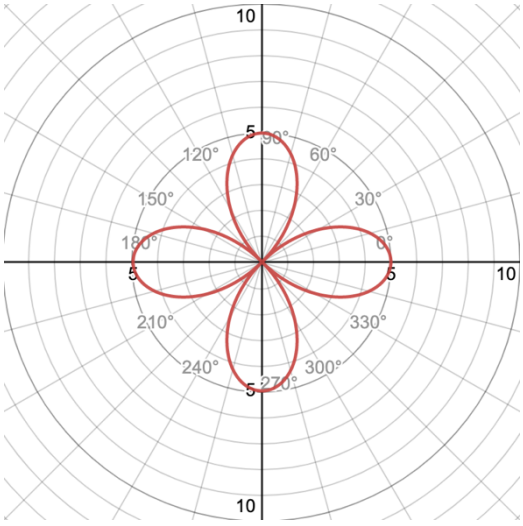
4. Given the polar curve $r = -5 \cos 2\theta$. (Answer i-ii).

i. Fill in the table below

$\boldsymbol{\theta}$	0	30	45	60	90	120	135	150	180
$\boldsymbol{r}$	-5	-2.5	0	2.5	5	2.5	0	-2.5	-5
$\boldsymbol{\theta}$	210	225	240	270	300	315	330		
$\boldsymbol{r}$	-2.5	0	2.5	5	2.5	0	-2.5		

ii. Sketch the graph of the function.



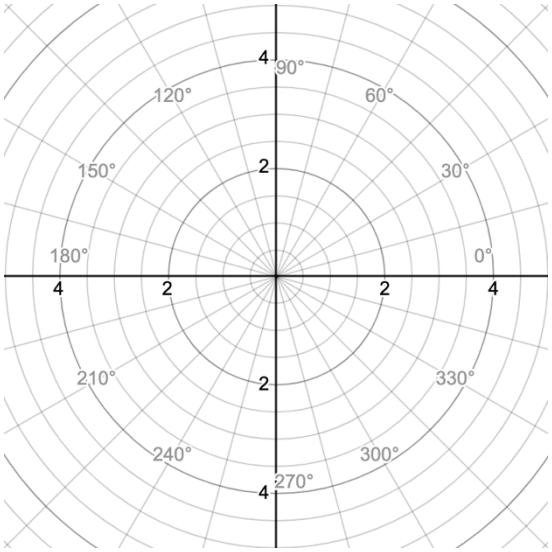


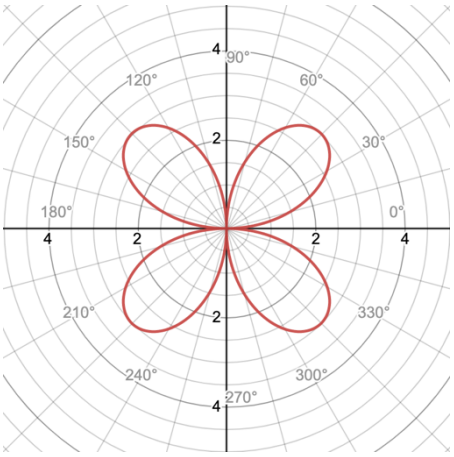
5. Given the polar curve $r = 3\sin 2\theta$. Answer questions (**i-ii**).

i. Fill in the table below

$\boldsymbol{\theta}$	0	30	45	60	90	120	135	150	180
$\boldsymbol{r}$	0	2.6	3	2.6	0	-2.6	-3	-2.6	0
$\boldsymbol{\theta}$	210	225	240	270	300	315	330		
$\boldsymbol{r}$	2.6	3	2.6	0	-2.6	-3	-2.6		

ii. Sketch the graph of the function.





Subtopic 1.3 Polar and Rectangular Forms of Equations

1. Find the **rectangular coordinates** of each point with the given polar coordinates.

i. $P(-2,\pi)$

$r=-2\,,\theta=\pi=180°$
$x=rcos\theta=-2cos180°=2$
$y=rsin\theta=-2sin180°=0$

The rectangular coordinates of P are $(2,0)$.

ii. $Q(3,\frac{5\pi}{6})$

$r=3\,,\theta=\dfrac{5\pi}{6}=150°$

$x=rcos\theta=3cos150°=\dfrac{-3\sqrt{3}}{2}\approx-2.6$

$y=rsin\theta=3sin150°=\dfrac{3}{2}=1.5$

The rectangular coordinates of Q are $(-2.6\,,1.5)$.

iii. $S(2\,,270°)$

$r=2\,,\theta=270°$
$x=rcos\theta=2cos270°=0$
$y=rsin\theta=2sin270°=-2$

The rectangular coordinates of S are $(0,-2)$

2. Which of the following represents the rectangular coordinates of the point $P(6, \frac{\pi}{6})$?

A. $(\sqrt{3}, 3)$

✓ B. $(3\sqrt{3}, 3)$

C. $(6, \pi)$

D. $(\sqrt{3}, 6)$

3. The rectangular form of the polar equation $r = 4\cos\theta$ is _____.

A. $x^2 + (y-2)^2 = 4$

B. $(x+2)^2 + y^2 = 4$

C. $x^2 + (y+2)^2 = 4$

✓ D. $(x-2)^2 + y^2 = 4$

4. Write the rectangular equation $(x-2)^2 + (y-8)^2 = 68$ in its polar form $r =$ ____, for $r \neq 0$.

A. $r = -6\cos\theta + 4\sin\theta$

✓ B. $r = 4\cos\theta + 16\sin\theta$

C. $r = 12\cos\theta + 16\sin\theta$

D. $r = 16\cos\theta - 4\sin\theta$

5. Given the rectangular coordinates for point $S(3, -3\sqrt{3})$, find the two pairs of polar coordinates for point S.

$S(x, y) = (3, -3\sqrt{3}), x = 3, y = -3\sqrt{3}$
$r = \sqrt{9 + 27} = \sqrt{36} = 6$
$x > 0, \theta = \tan^{-1}(-\sqrt{3})$
$\theta = -\frac{\pi}{3}, \frac{5\pi}{3}, \ldots.$
Polar coordinates are $\left(6, -\frac{\pi}{3}\right)$ *and* $(6, \frac{5\pi}{3})$

6. Convert the **polar equation to rectangular form** and identify the graph. Simplify your answer.

i. $r = 3sec\theta$

$r = \dfrac{3}{cos\theta}$
$rcos\theta = 3$
$x = 3$ vertical line

ii. $r = -2csc\theta$

$r = \dfrac{-2}{sin\theta}$
$rsin\theta = -2$
$y = -2$ horizontal line

iii. $r = -2sin\theta$

$r^2 = -2rsin\theta$
$x^2 + y^2 = -2y$
$x^2 + y^2 + 2y = 0$
$x^2 + (y+1)^2 - 1 = 0$
$x^2 + (y+1)^2 = 1$
the equation of the circle is $(x-h)^2 + (y-k)^2 = r^2$, whose center is (h, k) and radius r
Circle center $(0, -1)$, radius $\sqrt{1} = 1$

iv. $r = -4cos\theta$

$r^2 = -4rcos\theta$
$x^2 + y^2 = -4x$
$x^2 + 4x + y^2 = 0$
$(x+2)^2 - 4 + y^2 = 0$
$(x+2)^2 + y^2 = 4$
the equation of the circle is $(x-h)^2 + (y-k)^2 = r^2$, whose center is (h, k) and radius r
Circle center $(-2, 0)$, radius $\sqrt{4} = 2$

v. $rcsc\theta = 1$

$\frac{r}{sin\theta} = 1$
$rsin\theta = 1$
$y = 1$ horizontal line

vi. $r = 2sin\theta - 4cos\theta$

$r^2 = 2rsin\theta - 4rcos\theta$
$x^2 + y^2 = 2y - 4x$
$x^2 + 4x + y^2 - 2y = 0$
$(x+2)^2 - 4 + (y-1)^2 - 1 = 0$
$(x+2)^2 + (y-1)^2 = 5$

the equation of the circle is $(x-h)^2 + (y-k)^2 = r^2$, whose center is (h, k) and radius r
Circle center $(-2, 1)$, radius $\sqrt{5}$

7. Convert each **rectangular equation to polar form**. Find an expression of r.

i. $x = 2$

$rcos\theta = 2$
$r = \frac{2}{cos\theta}$
$r = 2sec\theta$

ii. $2x - 3y = 5$

$2(rcos\theta) - 3(rsin\theta) = 5$
$r(2cos\theta - 3sin\theta) = 5$
$r = \frac{5}{2cos\theta - 3sin\theta}$

iii. $(x-3)^2+y^2=9$

$x^2-6x+9+y^2=9$
$x^2+y^2-6x=0$
$r^2-6rcos\theta=0$
$r(r-6cos\theta)=0$
$r-6cos\theta=0$
$r=6cos\theta$

iv. $(x-1)^2+(y+4)^2=17$

$x^2-2x+1+y^2+8y+16=17$
$x^2+y^2-2x+8y=0$
$r^2-2rcos\theta+8rsin\theta=0$
$r(r-2cos\theta+8sin\theta)=0$
$r-2cos\theta+8sin\theta=0$
$r=2cos\theta-8sin\theta$

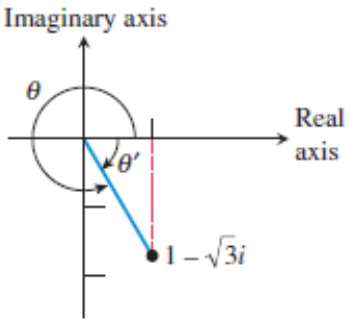
v. $y=-x^2$

$rsin\theta=-(rcos\theta)^2$
$rsin\theta=-r^2cos^2\theta$
$sin\theta=-rcos^2\theta$
$r=-\frac{sin\theta}{cos^2\theta}$
OR
$r=-tan\theta sec\theta$

Subtopic 1.5 Complex and De Moivre's Theorem

1. For each of the complex numbers given below, graph and find the absolute value. Also, express it in polar form.

i. $z = 1 - \sqrt{3}\,i$

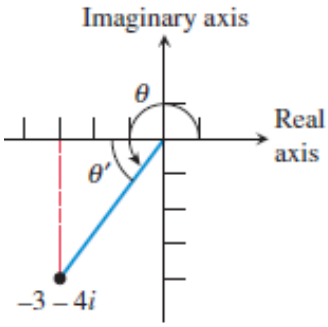


$$|z| = \sqrt{(1)^2 + (-\sqrt{3})^2} = 2$$

$$cos\theta = \frac{1}{2} > 0 \rightarrow \theta = \tan^{-1}\frac{-\sqrt{3}}{1} = \frac{-\pi}{3}\left(Or\ \frac{-\pi}{3} + 2\pi = \frac{5\pi}{3}\right)$$

$$z = 2(cos\frac{5\pi}{3} + i\ sin\frac{5\pi}{3})$$

ii. $z = -3 - 4\,i$



$$|z| = \sqrt{(-3)^2 + (-4)^2} = 5$$

$$cos\theta = \frac{-3}{5} < 0 \rightarrow \theta = \tan^{-1}\left(\frac{-4}{-3}\right) + 180° = 233.13° = \frac{233.13\pi}{180}\text{ rd}$$

$$z = 5(cos\frac{233.13\pi}{180} + i\ sin\frac{233.13\pi}{180})$$

2. Find $2(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}) \cdot 5(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9})$ in polar form. Then express the product in rectangular form.

$$10[\cos\left(\frac{2\pi}{9} + \frac{\pi}{9}\right) + i\sin\left(\frac{2\pi}{9} + \frac{\pi}{9}\right)]$$
$$10[\cos\left(\frac{3\pi}{9}\right) + i\sin\left(\frac{3\pi}{9}\right)]$$
$$10[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)]$$

$$10[\frac{1}{2} + \frac{\sqrt{3}}{2}i]$$
$$5 + 5i\sqrt{3}$$

3. Find $\frac{20(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4})}{4(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})}$ in polar form. Then express the product in rectangular form.

$\frac{20}{4}\left[\cos\left(\frac{3\pi}{4}-\frac{\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}-\frac{\pi}{4}\right)\right]$
$5\left[\cos\left(\frac{2\pi}{4}\right)+i\sin\left(\frac{2\pi}{4}\right)\right]$
$5\left[\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right]$
$5[0+1i]= 5i$

4. Find the product of z_1 and z_2. Express in rectangular form.

i. $z_1 = 25\sqrt{2}\,(cos\frac{-\pi}{4}+i\,sin\frac{-\pi}{4})\;\; z_2 = 14\,(cos\frac{\pi}{3}+i\,sin\frac{\pi}{3})$

$z_1z_2 = 350\sqrt{2}\,\left(cos\frac{\pi}{12}+i\,sin\frac{\pi}{12}\right) \approx 478.11 + 128.11\,i$

ii. $z_1 = 7\,(cos25° + i\,sin25°)\;\; z_2 = 2\,(cos130° + i\,sin130°)$

$z_1z_2 = 14\,(cos155° + i\,sin155°) \approx -12.68 + 5.91\,i$

5. Use an algebraic method to express the quotient of z_1 and z_2

$z_1 = 2\sqrt{2}\,(cos135° + i\,sin135°)\;\; z_2 = 6\,(cos300° + i\,sin300°)$

$\frac{z_1}{z_2} = \frac{\sqrt{2}}{3}\,(\cos(-165°) + i\sin(-165°)) \approx -0.46 - 0.12\,i$

6. Find $(1+i\sqrt{3})^3$

$z = 1 + i\sqrt{3}$
$r = \sqrt{(1)^2+(\sqrt{3})^2} = 2$

$cos\theta = \frac{1}{2} > 0 \rightarrow \theta = tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = 60°\; or\; \frac{\pi}{3}\; rd$

$z = 2(cos60° + isin60°)$

Using DeMoivre's Theorem:

$z^3 = (2)^3(cos180° + i\,sin180°) = 8(-1+0) = -8$

7. Find $(\frac{-\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})^8$

$z = \frac{-\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$r = \sqrt{(\frac{-\sqrt{2}}{2})^2 + (\frac{\sqrt{2}}{2})^2} = 1$

$cos\theta = \frac{-\sqrt{2}}{2} < 0 \rightarrow \theta = tan^{-1}(-1) + 180° = 135°$

$z = 1(cos135° + isin135°)$

Using DeMoivre's Theorem:

$z^8 = (1)^8(cos1080° + i\ sin1080°) = 1(1 + 0) = 1$

8. Find $(4 + 4\sqrt{3}i)^6$ and express it in a rectangular form.

$r = \sqrt{4^2 + (4\sqrt{3})^2} = 8$

$\theta = \tan^{-1}\frac{4\sqrt{3}}{4} = \frac{\pi}{3}(\cos\theta > 0)$

The polar form is $8(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$

Using DeMoivre's Theorem:

$(4 + 4\sqrt{3}i)^6 = [8\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)]^6 = 8^6\left[8\left(\cos\frac{6\pi}{3} + i\sin\frac{6\pi}{3}\right)\right] = 8^6(cos2\pi + isin2\pi) = 8^6.$

9. Express the complex number $2 - 2i$ in polar form.

A. $4\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

✓ B. $2\sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$

C. $2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

D. $3\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

10. Find the product $-6\left(\cos\frac{3\pi}{4}+\mathrm{i}\sin\frac{3\pi}{4}\right)\cdot 2\left(\cos\frac{2\pi}{3}+\mathrm{i}\sin\frac{2\pi}{3}\right)\cdot$

✓ A. $-12\left(\cos\frac{17\pi}{12}+\mathrm{i}\sin\frac{17\pi}{12}\right)$

B. $-3\left(\cos\frac{\pi}{12}+\mathrm{i}\sin\frac{\pi}{12}\right)$

C. $-12\left(\cos\frac{\pi}{12}+\mathrm{i}\sin\frac{\pi}{12}\right)$

D. $-3\left(\cos\frac{17\pi}{12}+\mathrm{i}\sin\frac{17\pi}{12}\right)$

11. Find the Quotient $-6\left(\cos\frac{3\pi}{4}+\mathrm{i}\sin\frac{3\pi}{4}\right)\div 2\left(\cos\frac{2\pi}{3}+\mathrm{i}\sin\frac{2\pi}{3}\right)\cdot$

A. $-12\left(\cos\frac{17\pi}{12}+\mathrm{i}\sin\frac{17\pi}{12}\right)$

✓ B. $-3\left(\cos\frac{\pi}{12}+\mathrm{i}\sin\frac{\pi}{12}\right)$

C. $-12\left(\cos\frac{\pi}{12}+\mathrm{i}\sin\frac{\pi}{12}\right)$

D. $-3\left(\cos\frac{17\pi}{12}+\mathrm{i}\sin\frac{17\pi}{12}\right)$

12. Find the below power and express it in rectangular form.

$\left[4\left(\cos\frac{\pi}{2}+i\cdot\sin\frac{\pi}{2}\right)\right]^4$

✓ A. 256

B. 64

C. $256i$

D. $256+16i$

13. Find the fourth roots of $z = 5(cos\frac{\pi}{3} + isin\frac{\pi}{3})$

The fourth roots of z are the complex numbers:

$$z = \sqrt[4]{5}\,(cos\frac{\frac{\pi}{3}+2k\pi}{4} + i\,sin\frac{\frac{\pi}{3}+2k\pi}{4}) \text{ for } k = 0,1,2,3$$

$$z_1 = \sqrt[4]{5}\left(cos\frac{\frac{\pi}{3}}{4} + i\,sin\frac{\frac{\pi}{3}}{4}\right) = \sqrt[4]{5}\left(cos\frac{\pi}{12} + i\,sin\frac{\pi}{12}\right) \quad \text{for } k = 0$$

$$z_2 = \sqrt[4]{5}\left(cos\frac{\frac{\pi}{3}+2\pi}{4} + i\,sin\frac{\frac{\pi}{3}+2\pi}{4}\right) = \sqrt[4]{5}\left(cos\frac{7\pi}{12} + i\,sin\frac{7\pi}{12}\right) \quad \text{for } k = 1$$

$$z_3 = \sqrt[4]{5}\left(cos\frac{\frac{\pi}{3}+4\pi}{4} + i\,sin\frac{\frac{\pi}{3}+4\pi}{4}\right) = \sqrt[4]{5}\left(cos\frac{13\pi}{12} + i\,sin\frac{13\pi}{12}\right) \quad \text{for } k = 2$$

$$z_4 = \sqrt[4]{5}\left(cos\frac{\frac{\pi}{3}+6\pi}{4} + i\,sin\frac{\frac{\pi}{3}+6\pi}{4}\right) = \sqrt[4]{5}\left(cos\frac{19\pi}{12} + i\,sin\frac{19\pi}{12}\right) \quad \text{for } k = 3$$

14. Find the cube roots of $z = -1$

$$z = (cos\pi + i\,sin\pi)$$

$$z = 1(cos\frac{\pi+2k\pi}{3} + i\,sin\frac{\pi+2k\pi}{3}) \text{ for } k = 0,1,2,3$$

$$z_1 = \left(cos\frac{\pi}{3} + i\,sin\frac{\pi}{3}\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2} \quad \text{for } k = 0$$

$$z_2 = \left(cos\frac{\pi+2\pi}{3} + i\,sin\frac{\pi+2\pi}{3}\right) = (cos\pi + i\,sin\pi) = -1 \quad \text{for } k = 1$$

$$z_3 = \left(cos\frac{\pi+4\pi}{3} + i\,sin\frac{\pi+4\pi}{3}\right) = \left(cos\frac{5\pi}{3} + i\,sin\frac{5\pi}{3}\right) = \frac{1}{2} - i\frac{\sqrt{3}}{2} \quad \text{for } k = 2$$

15. Answer questions (**i-iv**).

i. Write the complex number $z_1 = 4\left(cos\ \frac{5\pi}{6} + i\ sin\ \frac{5\pi}{6}\right)$ in rectangular form.

$a = r\cos\theta = 4\cos\frac{5\pi}{6} = -2\sqrt{3}$
$b = r\sin\theta = 4\sin\frac{5\pi}{6} = 2$
$z_1 = -2\sqrt{3} + 2i$

ii. Given $z_1 = 4\left(cos\ \frac{5\pi}{6} + i\ sin\ \frac{5\pi}{6}\right)$ and $z_2 = 2\left(cos\ \frac{\pi}{3} + i\ sin\ \frac{\pi}{3}\right)$, find the product $z_1 \cdot z_2$.

$z_1 \cdot z_2 = 4\left(cos\ \frac{5\pi}{6} + i\ sin\ \frac{5\pi}{6}\right) \cdot 2\left(cos\ \frac{\pi}{3} + i\ sin\ \frac{\pi}{3}\right)$
$= 4 \cdot 2\left(cos\ \left(\frac{5\pi}{6} + \frac{\pi}{3}\right) + i\ sin\ \left(\frac{5\pi}{6} + \frac{\pi}{3}\right)\right)$
$= 8\left(cos\ \left(\frac{7\pi}{6}\right) + i\ sin\ \left(\frac{7\pi}{6}\right)\right)$

iii. Use De Moivre's Theorem to find the **<u>fifth power</u>** of the complex number $z_2 = 2\left(cos\ \frac{\pi}{3} + i\ sin\ \frac{\pi}{3}\right)$.

$(z_2)^5 = 2^5\left(cos\ \left(5\left(\frac{\pi}{3}\right)\right) + i\ sin\ \left(5\left(\frac{\pi}{3}\right)\right)\right)$
$= 32\left(cos\ \frac{5\pi}{3} + i\ sin\ \frac{5\pi}{3}\right)$

iv. Find **<u>any one</u>** of the cube roots of the complex number $z_1 = 4\left(cos\ \frac{5\pi}{6} + i\ sin\ \frac{5\pi}{6}\right)$.

$\sqrt[3]{z_1} = \sqrt[3]{4}\left(\cos\frac{2\pi k + \frac{5\pi}{6}}{3} + i\sin\frac{2\pi k + \frac{5\pi}{6}}{3}\right)$
$k = 0, 1, 2$:
$Z_1 = \sqrt[3]{4}\left(\cos\frac{5\pi}{18} + i\sin\frac{5\pi}{18}\right)$
$Z_2 = \sqrt[3]{4}\left(\cos\frac{17\pi}{18} + i\sin\frac{17\pi}{18}\right)$
$Z_3 = \sqrt[3]{4}\left(\cos\frac{29\pi}{18} + i\sin\frac{29\pi}{18}\right)$

In this problem, it suffices to provide any of the cube roots.